Existence of Positive Solution for a higher-order fractional boundary value problem

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Abstract

This paper deals with the study of the existence of positive solutions for a class of nonlinear higher-order fractional differential equations. The results are established by applying Krasnoselsk'ii fixed point theorem and the well-known Guo-Krasnoselskii fixed point theorem in cone. Two examples are given to illustrate our results.

Keywords: Fractional differential equations, Fixed point theorem, existence solution, Positive solution.

Mathematics Subject Classifications: 34B10, 34B15.

1. Introduction

Fractional differential equations have recently proved to be the best tools in engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium.

The fractional calculus theory is a mathematical analysis tool applied to the study of integrals and derivatives of arbitrary order, which generalizes the classical notions of differentiation and integration. Many studies of fractional calculus and fractional differential equations have involved different derivatives such as Riemann--Liouville, Caputo, Hadamard and Grunwald--Letnikov,....

The theory of boundary value problems for nonlinear fractional differential equations belong to the important issues for the theory of fractional differential equations and need to be explored while numerous applications and physical manifestations of fractional calculus have been found and some existence results for nonlinear fractional boundary value problems was established by the use of techniques of nonlinear analysis such Banach fixed point theorem, Leray-Schauder theory, etc, see [2,4 – 10,13,15 – 20] .

By means of fixed point theorem for the mixed monotone operator, S. Zhang [21] studied the existence, multiplicity and nonexistence of positive solutions for the following higher order fractional boundary value problem

$$\begin{cases} {}^{c}D^{q}x(t)+\lambda h(t)f(u), \quad 0<t<1 \\ u(1)=u'(0)=\ldots=u^{m-2}(0)=u^{m-1}(0)=0, \end{cases} \qquad (1.1)$$

for $n \geq 2, \quad n-1<\alpha \leq n.$

B. Ahmed and Juan J. Nieto [1] , studied some existence results in Banach space for nonlocal boundary value problem involving a nonlinear differential equation of fractional order q given by

$$\begin{cases} {}^{c}D^{q}x(t)=f(t,u(t)), 0<t<1 \\ x(0)=0, x'(0)=0, x''(0)=0, \ldots, x^{(m-2)}(0)=0, x(1)=\beta x(\eta), \end{cases}$$

Where, $q \in (m-1, m]$, $m \in \mathrm{N}$, $m \geq 2$, ${}^{c}D$ is the Caputo fractional derivative and $f : [0,1] \times \mathrm{E} \to \mathrm{E}$ is continuous.

Motivated by the above mentioned works and and other works, we discuss in this paper the existence and the positivity of solutions for the following nonlinear higher-order fractional boundary value problem

$$\begin{cases} D_{0^{+}}^{\alpha}u(t)+f(t,u(t))=0, \quad t \in (0,1). \\ u(0)=u'(0)=u''(0)=\ldots=u^{(n-2)}(0)=0, \quad u(1)=\beta u(\eta), \end{cases}$$

where: (i) $f \in C([0,1] \times \mathrm{R}^{+}, \mathrm{R}^{+})$, 😐☹0, 0 🖫☼🖫1.

(ii) $D_{0^{+}}^{\alpha}$ is the standar Riemann-Liouville fractional derivative of order $n-1<\alpha \leq n, \quad n \geq 3.$

The remaining part of the paper is organized as follows. In Section 2, we recall some basic properties and introduce some new lemmas which will be used later. the existence of solutions is obtained in Section 3. In section4, we give some properties of Green's function, and then we study the existence of positive solutions. In Section 5, two examples are given to demonstrate the application of our main results.

1. Preliminaries

Let us recall some basic definitions on fractional calculs and some important preliminary lemma.

Let X be the Banach space of continuous functions $C[0,1]$, endowed with the norm $\|u\|_{X} = \max_{t \in [0,1]} |u(t)|.$

Definition *1.* *The fractional integral*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where $\alpha > 0$ *, is called Riemann-Liouville fractional integral of order* ج *of a function* $f : (0,+\infty) \to \mathrm{R}$ *and* $\Gamma(.)$ *is the gamma function defined by*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-s} ds.$$

Definition 2. *The Riemann-Liouville fractional derivative of order* $\alpha > 0$ *, of a continuous function* $f : (0,+\infty) \to \mathrm{R}$ *is given by*

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds.$$

$\Gamma(.)$ *is the gamma function, provided that the right side is point-wise defined on* $(0,+\infty)$ *and* $n = [\alpha]+1,$ $[\alpha]$ *stands for the greatest integer less than* α.

Lemma 3. [11] *Let* $\alpha, \beta \geq 0,$ $f \in L^1(0,1),$ *then*

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = f(t), \quad I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t).$$

The following two lemmas can be found in [9,14].

Lemma 4. [11] *Let* $\alpha > 0$ *and* $u \in C(0,1) \cap L^1(0,1),$ *then fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0,$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}, \quad c_i \in \mathrm{R},\ i = 1,2,\ldots,n;\ n = [\alpha]+1,$$

as solution.

Lemma 5. [12] *Assume that* $u \in C(0,1) \cap L^1(0,1)$ *with a frational derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L^1(0,1).$ *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

for some $c_i \in \mathrm{R},$ $i = 1,2,\ldots,n;$ $n = [\alpha]+1.$

Lemma 6. [14] *For Riemann-Liouville fractional derivatives, we have*

$$D_{0^+}^{\beta} \int_0^t (t-s)^{\alpha-1} f(s) ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s) ds,$$

where $f \in C[0,1],$ $\alpha,$ β *are two constants with* $\alpha > \beta \geq 0.$

Lemma 7. *Let* $\beta\eta^{\alpha-1} \neq 1$ *and* $y \in L^1[0,1],$ *then the problem*

$$D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1,$$

$$u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \quad u(1) = \beta u(\eta),$$

has a unique solution

$$u(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1} y(s)ds$$

$$-\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^\eta (\eta-s)^{\alpha-1} y(s)ds.$$

Proof. We have

$$u(t)=-I_{0^+}^{\alpha} y(t)+C_1 t^{\alpha-1}+C_2 t^{\alpha-2}+....+C_n t^{\alpha-n}.$$

From $u(0)=0$ we get $C_n=0,$

$$u(t)=-I_{0^+}^{\alpha} y(t)+C_1 t^{\alpha-1}+C_2 t^{\alpha-2}+....+C_{n-1} t^{\alpha-n+1}.$$

Thus,

$$u'(t)=-\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2} y(s)ds+C_1(\alpha-1)t^{\alpha-2}+....+C_{n-1}(\alpha-n+1)t^{\alpha-n}.$$

And from $u'(0)=0$ we get $C_{n-1}=0.$ Similary, from $u''(0)=...=u^{(n-2)}(0)=0,$ we get $C_1=C_2=...=C_{n-2}=0.$ Then

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)ds+C_1 t^{\alpha-1}.$$

From $u(1)=\beta u(\eta),$ we deduce that

$$C_1=\frac{1}{1-\beta\eta^{\alpha-1}}\left[I_{0^+}^{\alpha} y(1)-\beta I_{0^+}^{\alpha} y(\eta)\right]$$

Then

$$u(t)=\frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1} y(s)ds$$

$$-\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^\eta (\eta-s)^{\alpha-1} y(s)ds.$$

The proof is complete.

2. Existence results

Now we state a known result due to Krasnoselskii, which is needed.

Theorem 8. [3] (*Krasnoselskii fixed point theorem*) *Let* X *be a closed convex and nonempty subset of a Banach space* E *. Let* A *and* B *be two operators such that*

1. $Ax+By\in X$, *whenever* $x,y\in X$.
2. A *is compact and continuous.*
3. B *is a contraction.*

Then there exists $z\in X$ *such that* $z=Az+Bz.$

Theorem 9. *Assume that* $\beta\eta^{\alpha-1}\neq 1$ *and there exists a nonnegative function* $\rho(t)\in L^1(0,1)$ *such that*

$$|f(t,u)-f(t,v)|\leq \rho(t)\|u-v\|,\ \forall u,v\in \mathrm{R}^+,\ t\in[0,1]=I.$$

Then problem (1.1) *has at least one solution on* X.

Proof. Let $\chi=\{f(t,0),\ t\in[0,1]\}$ Consider the set $B_r=\{u\in X\ :\ \|u\|\leq r\}$, then B_r is a closed, bounded, and convex set of X . We define the operators A and B on X as

$$Au(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds.$$

$$Bu(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1} f(s,u(s))ds$$
$$-\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^\eta (\eta-s)^{\alpha-1} f(s,u(s))ds.$$

For any $u\in B_r$ and $t\in I$, we get with the help of inequality (3.1)

$$|Au(t)|\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))|ds.$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u(s))-f(s,0)|\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,0)|\,ds.$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)|u(s)|\,ds+\frac{\chi}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds.$$

Hence, we get

$$\|Au\|\leq \frac{\rho^*}{\Gamma(\alpha)}\|u\|+\frac{\chi}{\Gamma(\alpha)},$$

where, $\rho^*=\int_0^1 \rho(s)ds.$

Similarly, we estimate $\|Bv\|$. Let $v\in B_r$ and $t\in I$, then

$$|Bv(t)|\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^1 (1-s)^{\alpha-1}|f(s,v(s))|\,ds$$
$$+\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\int_0^\eta (\eta-s)^{\alpha-1}|f(s,v(s))|\,ds.$$

$$\leq \frac{t^{\alpha-1}\rho^*}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\|v\| + \frac{t^{\alpha-1}\chi}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}$$
$$+\frac{t^{\alpha-1}\beta\rho^*}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\|v\| + \frac{t^{\alpha-1}\beta\chi}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}$$

$$\|Bv\| \leq \frac{(1+\beta)\rho^*}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\|v\| + \frac{(1+\beta)\chi}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}.$$

Taking estimates (3.2) and (3.3) into account, we get for any $u,v \in B_r$ and $t \in I$,

$$\|Au+Bv\| \leq \|Au\| + \|Bv\|.$$
$$\leq \frac{\rho^*}{\Gamma(\alpha)}\|u\| + \frac{\chi}{\Gamma(\alpha)} + \frac{(1+\beta)\rho^*}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\|v\| + \frac{(1+\beta)\chi}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)},$$
$$\leq \frac{\rho^*}{\Gamma(\alpha)}\|u\| + \frac{(1+\beta)\rho^*}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\|v\| + \frac{\chi}{\Gamma(\alpha)} + \frac{(1+\beta)\chi}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)},$$
$$\leq r\left[\rho^* + \frac{(1+\beta)\rho^*}{\left(1-\beta\eta^{\alpha-1}\right)}\right] + \chi\left(1 + \frac{1+\beta}{\left(1-\beta\eta^{\alpha-1}\right)}\right).$$

Since if

$$r \geq \frac{\chi\left(2-\beta\left(\eta^{\alpha-1}-1\right)\right)}{\left(1-\rho^*\right)\left(1-\beta\eta^{\alpha-1}\right)+(1+\beta)\rho^*}.$$

Then,

$$\|Au+Bv\| \leq r.$$

Now, we prove that B is a contraction. Let $v,u \in B_r$, and $t \in I$. Then, thanks to (3.1), it yields

$$|Bu(t)-Bv(t)| \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\int_0^1 (1-s)^{\alpha-1}\left|f(s,u(s))-f(s,v(s))\right| ds$$
$$+\frac{t^{\alpha-1}\beta}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f(s,u(s))-f(s,v(s))\right| ds.$$

$$\leq \frac{1}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\int_0^1 (1-s)^{\alpha-1}\rho(s)|u(s)-v(s)|\, ds$$
$$+\frac{\beta}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\int_0^\eta (\eta-s)^{\alpha-1}\rho(s)|u(s)-v(s)|\, ds.$$

$$\leq \frac{\|u-v\|}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\int_0^1 \rho(s)ds$$
$$+\frac{\beta}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)}\int_0^\eta \rho(s)ds.$$

$$\leq \frac{(1+\beta)\rho^*}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}\|u-v\|.$$

so, if $\frac{(1+\beta)\rho^*}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \leq 1,$ we conclude that B is a contraction.

Let us prove that A is compact and continuous. The continuity of f implies that A is continuous. Also A is uniformly bounded on B_r indeed, from (3.3) we have

$$\|Au\| \leq \frac{\rho^*}{\Gamma(\alpha)}\|u\| + \frac{\chi}{\Gamma(\alpha)},$$
$$\leq \frac{\rho^*}{\Gamma(\alpha)} r + \frac{\chi}{\Gamma(\alpha)}.$$

Set $L = \max\{|f(s,u(s))|, u \in B_r\}$. Let $u \in B_r$, $t_1, t_2 \in I$, with $t_1 \leq t_2$. We have

$$|Au(t_2) - Au(t_1)| \leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1} f(s,u(s))ds \right.$$
$$\left. - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}|f(s,u(s))|ds\right|.$$

$$|Au(t_2) - Au(t_1)| \leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] f(s,u(s))ds\right|$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s,u(s))|ds.$$

$$\leq \frac{L}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]ds$$
$$+ \frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds.$$

$$= \frac{L}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}.$$

Hence, if $t_2 \to t_1$, then $|Au(t_2) - Au(t_1)| \to 0$. Then A is equicontinuous and so, by Arzela-Ascoli theorem, we deduce that A is compact on B_r . So the operator A is completely continuous. Thus, by *Theorem* 8 , problem (1.1) has at least one solution in X . The proof is complete.

3. Positive results

Now we state a known result due to Guo-Krasnoselskii, which is needed to prove the existence of positive solution to the posed problem .

Theorem 10. [9] *Let* E *be a Banach space, and let* $K \subset E$, *be a cone. Assume* Ω_1, Ω_2 *are open subsets of* E *with* $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, *and let*

$$\mathbf{A} : K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right) \to K,$$

be a completely continuous operator. In addition suppose either

(i) $\|\mathbf{A}u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathbf{A}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|\mathbf{A}u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|\mathbf{A}u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$,

holds. Then $\mathbf{A}$ *has a fixed point in* $K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$.

Lemma 11. *The solution of boundary value problem* (1.1) *can be expressed as*

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \int_0^1 G(\eta,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. Using *Lemma* 7, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left[-(t-s)^{\alpha-1} + t^{\alpha-1}(t-s)^{\alpha-1}\right] y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_t^1 (1-s)^{\alpha-1} y(s)ds$$

$$+ \frac{t^{\alpha-1}\beta}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \int_0^\eta \left[\eta^{\alpha-1}(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right] y(s)ds$$

$$+ \frac{t^{\alpha-1}\beta}{\Gamma(\alpha)\left(1-\beta\eta^{\alpha-1}\right)} \int_\eta^1 \eta^{\alpha-1}(1-s)^{\alpha-1} y(s)ds.$$

And, that is equivalent to

$$u(t) = \int_0^1 G(t,s)y(s)ds + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \int_0^1 G(\eta,s)y(s)ds.$$

The proof is complete.

Lemma 12. $G(t,s)$ *is strictly increasing in the first variable.*

Proof. In the case, $s \le t$:

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left[t^{\alpha-1}(t-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] = G_1(t,s).$$

In the case, $t \le s$:

$$G(t,s) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(t-s)^{\alpha-1} = G_2(t,s).$$

It is easy to check that $G_1(t,s)$ is strictly increasing on $[s,1]$ and, $G_2(t,s)$ is strictly increasing on $[0,s]$.

For: $t_1, t_2 \le s$ and $t_1 < t_2$, we have $G_2(t_1, s) < G_2(t_2, s)$.
For: $s \le t_1, t_2$ and $t_1 < t_2$, we have $G_1(t_1, s) < G_1(t_2, s)$.
Then, we obtain: $G(t_1, s) < G(t_2, s)$.
In the case, $t_1 \le s \le t_2$ and $t_1 < t_2$, we have

$$G_2(t_1, s) \le G_2(s, s) = G_1(s, s) < G_1(t_2, s)$$

We claim that

$$G_2(t_1, s) < G_1(t_2, s).$$

In fact, if $G_2(t_1, s) = G_1(t_2, s)$ then, $G_2(t_1, s) = G_2(s, s) = G_1(s, s) = G_1(t_2, s)$, and from the monotonicity of G_1 and G_2, we have $t_1 = s = t_2$, which contradicts with $t_1 < t_2$. This fact implies that $G(t_1, s) < G(t_2, s)$. The proof is complete.

Lemma 13. *The function* $G(t,s)$ *defined by* (3.2) *satisfies the following properties*

(i) $G(t,s) \ge 0$ *and* $G(t,s) \in C([0,1] \times [0,1], \mathbb{R}^+)$.

(ii) *If* $t, s \in [\tau, 1]$, $\tau > 0$, *then*

$$\tau^{\alpha-1} G_1(s) \le G(t,s) \le \frac{1}{\tau} G_1(s),$$

where $G_1(s) = \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-1}$.

Proof. (i) The continuity of G is easily checked. For $0 \le t \le s \le 1$, it is obvious that

$$G(t,s) = \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} \ge 0.$$

In the case, $0 \le s \le t \le 1$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1} \right] = \frac{(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \ge 0.$$

(ii)

If $0 \le t \le s \le 1$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1} t^{\alpha-1} \le G_1(s).$$

If $0 \le s \le t \le 1$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1} \right],$$

then

$$G(t,s) \le \frac{1}{s} G_1(s), \quad \forall s,t \in [0,1].$$

Consequently

$$G(t,s) \le \frac{1}{\tau} G_1(s), \quad \forall s \in [\tau, 1], \; t \in [0,1].$$

Now we look for lower bounds of $G(t,s)$. If $0 \le t \le s \le 1$,

$$G(t,s) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-1} \ge \frac{1}{\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1},$$

then,

$$G(t,s) \ge t^{\alpha-1} G_1(s), \quad \forall s,t \in [0,1].$$

If $0 \le s \le t \le 1$, we have

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1} t^{\alpha-1} - (t-s)^{\alpha-1}\right] \ge 0,$$

$$G(t,s) \ge t^{\alpha-1} G_1(s), \quad \forall s,t \in [0,1].$$

Consequently,

$$G(t,s) \ge \tau^{\alpha-1} G_1(s), \text{ for } t,s \in [\tau,1].$$

The proof is complete.

In this section, we discuss the existence of positive solution for *fractional boundary value problem* (1.1). We make the following additional assumptions.

(Q_1) $f(t,u) = a(t) f_1(u)$ where $a \in C((0,1), \mathbb{R}^+)$ and $f_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$.

(Q_2) $\int_0^1 G_1(s) a(s) ds > 0.$

Definition 14. *We define an operator* $T : E \to E$ *by*

$$Tu(t) = \int_0^1 G(t,s) a(t) f_1(u(s)) ds + \frac{\beta t^{\alpha-1}}{1-\beta\eta^{\alpha-1}} \int_0^1 G(\eta,s) a(t) f_1(u(s)) ds, \ t \in [0,1]. \tag{3.1}$$

The function $u \in X$ is a solution of the BVP (1.1) if and only if $Tu(t) = u(t)$, (u is a fixed point of T).

Lemma 15. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Lemma 16. *Let* $u \in X$, *the unique solution* u *of the fractional boundary value problem* (1.1) *is nonnegative and satisfies*

$$\min_{t \in [\tau,1]} u(t) \ge \tau^{\alpha} \|u\|_X .$$

Proof. Let $u \in X$, $t \in [0,1]$, it is obvious that $u(t)$ is nonnegative. We have

$$u(t) \le \frac{1}{\tau}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \int_0^1 G_1(s) a(s) f_1(u(s)) ds.$$

It yields

$$\|u\|_X \le \frac{1}{\tau}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right) \int_0^1 G_1(s) a(s) f_1(u(s)) ds.$$

Hence

$$\tau\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)^{-1}\|u\|_X \leq \int_0^1 G_1(s)a(s)f_1(u(s))ds.$$

On the other hand, for all $t\in[\tau,1]$, we obtain

$$u(t)\geq \tau^{\alpha-1}\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s)a(s)f_1(u(s))ds.$$

Therefore, we have

$$\min_{t\in[\tau,1]} u(t)\geq \tau^{\alpha}\|u\|_X .$$

The proof is complete.

Definition 17. *We define the cone* K *by*

$$K=\left\{u\in X,\ u(t)\geq 0,\ \min_{t\in[\tau,1]} u(t)\geq \tau^{\alpha}\|u\|_X\right\}$$

K is a non-empty closed and convex subset of X.

Lemma 18. [14] *The operator defined in* (4.1) *is completely continuous and satisfies* $T(K)\subseteq K$.

The main result of this section is the following

Theorem 19. *Let* (Q_1) *and* (Q_2) *hold,* $0<\beta\eta^{\alpha-1}<1$ *and assume that*

$$f_0=\lim_{|u|\to 0}\frac{f_1(u)}{|u|},\quad f_\infty=\lim_{|u|\to\infty}\frac{f_1(u)}{|u|}.$$

Then problem (1.1) *has at least one positive solution in the case*

(i) $f_0=0$ *and* $f_\infty=\infty$ (superlinear) *or*

(ii) $f_0=\infty$ *and* $f_\infty=0$ (sublinear).

Proof. *We shall prove that problem* BVP (1.1) *has at least one positive solution in both cases, superlinear and sublinear, for this we use* Theorem 10 . *We prove the superlinear case. Since* $f_0=0$, *then for any* $\varepsilon>0$, $\exists\delta_1>0$, *such that* $f_1(u)\leq\varepsilon|u|$, *for* $|u|<\delta_1$. *Let* Ω_1 *be an open set in* E *defined by*

$$\Omega_1=\left\{y\in X / \|y\|<\delta_1\right\},$$

then, for any $u\in K\cap\partial\Omega_1$, *it yields*

$$Tu(t)\leq\frac{1}{\tau}\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s)a(s)f_1(u(s))ds.$$

Therefore

$$\|Tu\|_X\leq\varepsilon\frac{1}{\tau}\|u\|_X\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s)a(s)ds,$$

If we choose $\varepsilon=\left[\frac{1}{\tau}\left(1+\frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s)a(s)ds\right]^{-1}$, *then it yields*

$$\|Tu\|_X \le \|u\|_X, \quad \forall u \in K \cap \partial\Omega_1.$$

Now from $f_\infty = \infty$, *then* $\forall M > 0$, $\exists H > 0$, *such that* $f_1(u) \ge M|u|$ *for* $|u| \ge H$. *Let*

$$H_1 = \max\left\{2\delta_1, \frac{\gamma}{\mu} H\right\}.$$

Denote by Ω_2 *the open set*

$$\Omega_2 = \{y \in X / \|y\| < H_1\}.$$

For any $u \in K \cap \partial\Omega_2$, *have*

$$\min_{t\in[\tau,1]} u(t) \ge \tau^\alpha \|u\|_X,$$

$$= \tau^\alpha H_1 \ge H,$$

alors, $\overline{\Omega_1} \subset \Omega_2$. *Let* $u \in K \cap \partial\Omega_2$ *then*

$$Tu(t) \ge \tau^{\alpha-1} \int_0^1 \left(1 + \frac{\beta}{1 - \beta\eta^{\alpha-1}}\right) G_1(s) a(s) f_1(u(s)) ds,$$

$$Tu(t) \ge \tau^{\alpha-1} \left(1 + \frac{\beta}{1 - \beta\eta^{\alpha-1}}\right) M \int_0^1 G_1(s) a(s) ds \|u\|_E,$$

And choosing $M = \left[\tau^{\alpha-1}\left(1 + \frac{\beta}{1-\beta\eta^{\alpha-1}}\right)\int_0^1 G_1(s) a(s) ds\right]^{-1}$, *we get*

$$\|Tu\|_X \ge \|u\|_X, \forall u \in K \cap \partial\Omega_2.$$

By the first part of Theorem 10 , T *has at least one fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, *such that;* $H \le \|y\| \le H_1$. *This completes the superlinear case of* Theorem 19 . ***Case II*** *Now, we assume that* $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear case)* . *Proceding as above and by the second part of* Theorem 10, *we prove the sublinear case. This achieves the proof of* Theorem 19 .

4. Examples

In order to illustrate our results, we give the following examples

Example 20. *Consider the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t) + \frac{t^{\frac{1}{2}} e^{-t}}{4} \frac{1}{1+u^2} = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \ u(1) = \beta u(\eta), \end{cases}$$

We have $f(t,u) = \frac{t^{\frac{1}{2}} e^{-t}}{4}$, *so* $\chi = 0.31$.

We can verify that

$$|f(t,u) - f(t,v)| \le \frac{t^{\frac{1}{2}} e^{-t}}{4} |u - v|.$$

Then, $\rho(t) = \frac{t^{\frac{1}{2}} e^{-t}}{4}$, *so* $\rho^* = \frac{0.378}{4}$, *and* $\frac{(1+\beta)\rho^*}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} < 1$.

Therefore, by Theorem 9 , *problem* (P_1) *has at least one solution in* B_r *with*

$r \geq \frac{\chi\left(2-\beta\left(\eta^{\alpha-1}-1\right)\right)}{\left(1-\rho^{*}\right)\left(1-\beta\eta^{\alpha-1}\right)+(1+\beta)\rho^{*}}.$

Example 21. *Consider the following fractional boundary value problem*

$$\begin{cases} D_{0^+}^{\frac{7}{2}} u(t)+t^2u^2e^{(1+u(t))}=0, \quad 0<t<1, \\ u(0)=u'(0)=u''(0)=0, \; u(1)=\beta u(\eta), \end{cases}$$

where, $0<\beta\eta^{\alpha-1}<1$, $n=4$, $\alpha=\frac{7}{2}$, *and*

$$f(t,u)=t^2\left(u^2e^{(1+u(t))}\right)=a(t)\,f_1(u),$$

$a(t)=t^2 \in C((0,1),\mathrm{R}^+)$, $f_1(u)\in C(\mathrm{R}^+,\mathrm{R}^+)$. *Then*

$$f_0=\lim_{|u|\to 0}\frac{f_1(u)}{|u|}=0, \quad f_\infty=\lim_{|u|\to 0}\frac{f_1(u)}{|u|}=\infty.$$

By Theorem 19 (i), *the fractional boundary value problem* (P_2) *has at least one positive solution.*

Conclusion 22. *In the present work, we have studied the existence and the positivity of solutions for a higher-order fractional boundary value problem. To demonstrate the existence results, we transformed the posed problem into a sum of a contraction and a compact operator, then we applied the Krasnoselskii's fixed point theorem. To prove the positivity results, we expressed the Green function associated to the posed problem, then we apply the well-known Guo-Krasnoselskii fixed point theorem in cone. We ended the article with two examples illustrating the obtain results.*

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