Certain Definite Integrals Involving Multivariable H-Functions

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ABSTRACT
In this paper, the author presented certain integrals involving product of the multivariable H-function with exponential function, Gauss's hypergeometric function and Fox's function. The results derived here and basic in natural and many include a number of known and new results as particular cases.
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I. INTRODUCTION

The Gaussian hypergeometric function is of fundamental important in the theory of special functional. The important of this function lies in the first that most all of the commonly used function of applicable mathematics, mathematical physics, engineering and mathematical biology are expressible as its special cases.
The series

$${}_2F_1(a,b;c:z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^u}{u!} \tag{1}$$

Where $(a)_n$ is the pochhammer symbol defined by

$$(a)_n=\begin{cases}a\,(a+1)\ldots\ldots(a+n-1) & n\in N\\ 1 & n=0\end{cases} \tag{2}$$

is called the Gauss's hypergeometric series of the famous German mathematician Carl Friedrich Gauss (1777-1855) who is the year 1812 introduced the series. It is represented by the symbol ${}_2F_1(a,b;c:z)$ and is called the Gauss's hypergeometric functional so.

In 1961 Charles Fox [2] introduced a function which is more general then the Meijer's G-function and this function is well known in the literature of special function as Fox's H-function or simply the H-function. This function is defined and represented by means of the following Mellin-Barnes type contour integral

$$\mathrm{H}[\mathrm{z}]=H_{p,q}^{m,n}[\mathrm{z}]=H_{p,q}^{m,n}\left[z\ \middle|\ \begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]=\frac{1}{2\pi i}\int_L\theta(s)z^s\,ds \tag{3}$$

Where for convenience

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} \tag{4}$$

and z is a suitable contour of the Mellin-Barnes type which runs from $-i\infty\ to\ +i\infty$, separating the poles of $\Gamma(b_j-\beta_j s)$, (j =1,… …,m) from those of $\Gamma(1-a_j+\alpha_j s)$,(j =1,… …,n). An empty product is interpreted asunity the integers m,n,p,q satisfy the inequalities $0\le n\le p, 0\le m\le q$; the coefficients $\alpha_j=(j=1,\ldots\ldots,\mathrm{p}),\beta_j=(j=1,\ldots\ldots,\mathrm{q})$ are positive real numbers, and the complex parameter $a_j=(j=1,\ldots\ldots,\mathrm{p}),b_j=(j=1,\ldots\ldots,\mathrm{q})$ are so constrained that no poles of the integrand coincide. Owing to the popularity of the special functions, those are defined in (1) and (3) (c.f [4],[3] and [5]), details regarding these are avoided.

The multivariable *H*-function which was introduced and investigated by Srivastava& Panda [5] in term of a multiple Mellin-Bernes type contour integral as

$$H[z_1,\ldots,z_r]$$

$$= H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j;\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,\ldots,\beta_j^r)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1,\ldots,\xi_r)\left\{\prod_{i=1}^{r}\phi_i(\xi_i)z_i^{\xi_i}\right\}d\xi_1\ldots d\xi_r, \tag{5}$$

Where $\omega=\sqrt{-1}$; and

$$\psi(\xi_1,\ldots,\xi_r)=\frac{\prod_{j=1}^{n}\Gamma(1-a_j+\Sigma_{i=1}^{r}\alpha_j^i\xi_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\Sigma_{i=1}^{r}\alpha_j^i\xi_i)\prod_{j=1}^{q}\Gamma(1-b_j+\Sigma_{i=1}^{r}\beta_j^i\xi_i)} \tag{6}$$

$$\phi_i(\xi_i)=\frac{\prod_{j=1}^{n_i}\Gamma(1-c_j^i+\gamma_j^i\xi_i)\prod_{j=1}^{m_i}\Gamma(d_j^i-\delta_j^i\xi_i)}{\prod_{j=n_i+1}^{p_i}\Gamma(c_j^i-\gamma_j^i\xi)_i\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^i+\delta_j^i\xi_i)}\quad (i=1,\ldots,r); \tag{7}$$

And $L_j=L_{\omega\tau_j\infty}$ represents the contours which start at the point $\tau_j-\omega\infty$ and terminate at the points $\tau_j+\omega\infty$ with $\tau_j\in\Re=(-\infty,\infty)(j=1,\ldots,r)$.

In case r = 2, (5) reduce to the H-function of two variables.
In case r = 1, (5) reduce to the H-function of one variables.

II. RESULT REQUIRED IN THE SEQUEL

We shall require the following results in the sequel :

$${}_2F_1(a,b,c+\frac{1}{2};X)\ {}_2F_1(c-a,c-b,\ c+\frac{1}{2};X)=\sum_{k=0}^{\infty}\frac{(c,k)}{\left(c+\frac{1}{2},k\right)}a_kX^k \tag{8}$$

and

From Rainville{1}:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{9}$$

III. MAINRESULTS

In this section we have evaluated certain integrals involving product of the Multivariable H-function with exponential function, Gauss's hypergeometric function and Fox's H-function.

Theorem1:

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{-zx}\,{}_2F_1\left(a,b,c+\tfrac{1}{2};x^\zeta(t-x)^\eta\right){}_2F_1\left(c-a,c-b,c+\tfrac{1}{2};x^\zeta(t-x)^\eta\right)$$

$$H^{0,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} y_1x^{\mu_1}(t-x)^{\upsilon_1} \\ \vdots \\ y_rx^{\mu_r}(t-x)^{\upsilon_r}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j,\beta_j^1,\ldots,\beta_j^r)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right.\right]dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u}$$

$$\times H^{0,n+2:m_1,n_1;...;m_r,n_r}_{p+2,q+1:p_1,q_1;...;p_r,q_r}\left[\begin{matrix} y_1 t^{(\mu_1+\upsilon_1)} \\ \vdots \\ y_r t^{(\mu_r+\upsilon_r)} \end{matrix}\left| \begin{matrix} (1-\rho-\zeta k;\prod_{i=1}^{r}\mu_i),(1-\sigma-(\eta-1)k-u;\prod_{i=1}^{r}\upsilon_i)(a_j;\alpha_j^1,...,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};...;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j;\beta_j^1,...,\beta_j^r)_{1,q},(1-\rho-\sigma-(\zeta+\eta-1)k-u;\prod_{i=1}^{r}\mu_i+\upsilon_i):(d_j^1,\delta_j^1)_{1,q_1};...;(d_j^r,\delta_j^r)_{1,q_r} \end{matrix}\right.\right] \quad (10)$$

$$f(k) = \sum_{k=0}^{\infty} \frac{(c,k)}{\left(c+\frac{1}{2},k\right)} a_k\, x^k (11)$$

Provided

(i) $\mu \geq 0, \nu \geq 0$ (not both zero simultaneously)

(ii) ς and η are non negative integers such that $\varsigma + \eta \geq 1$

(ii) $A_i > 0, \beta_i < 0 : |\arg y| < \frac{1}{2} A_i \pi \ \forall i \in 1,....,r$ *where*

$$A_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji}$$

$$\beta_i = \frac{1}{2}(p_i - q_i) + \sum_{j=1}^{q_i} b_{ji} - \sum_{j=1}^{p_i} \alpha_{ji}$$

$$\mathrm{Re}(\rho) + \mu \max_{0 \leq j \leq m} [\mathrm{Re}(b_j / \beta_j] > 0$$

$$\mathrm{Re}(\sigma) + \nu \max_{0 \leq j \leq m} [\mathrm{Re}(b_j / \beta_j] > 0$$

Proof:

$$e^{-zt} \int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx} {}_2F_1\left(a,b,c+\tfrac{1}{2};x^{\zeta}(t-x)^{\eta}\right) {}_2F_1\left(c-a,c-b,c+\tfrac{1}{2};x^{\zeta}(t-x)^{\eta}\right)$$

$$\times H^{0,n;m_1,n_1;...;m_r,n_r}_{p,q;p_1,q_1;...;p_r,q_r}\left[\begin{matrix} y_1 x^{\mu_1} (t-x)^{\upsilon_1} \\ \vdots \\ y_r x^{\mu_r} (t-x)^{\upsilon_r} \end{matrix}\left| \begin{matrix} (a_j,\alpha_j^1;...;\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};...;(c_j^r,\gamma_j^r)_{1,p_r} \\ (b_j,\beta_j^1;...;\beta_j^r)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};...;(d_j^r,\delta_j^r)_{1,q_r} \end{matrix}\right.\right] dx$$

[Now we replace $e^{(t-x)z}$ with $\sum_{u=0}^{\infty} \frac{(t-x)^u z^u}{u!}$ and express the hypergeometric function and Multivariable H-function with the help of (5) and (8) respectively, to get]

$$= e^{-zt} \int_0^t x^{\rho-1}(t-x)^{\sigma-1} \sum_{u=0}^{\infty} \frac{(t-x)^u z^u}{u!} \sum_{k=0}^{\infty} \frac{(c,k)}{\left(c+\frac{1}{2},k\right)} a_k\, x^{\zeta k}(t-x)^{\eta k}$$

$$\times \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \quad \emptyset_1(\xi_1) \ldots \emptyset_r(\xi_r) \psi(\xi_1, \ldots, \xi_r)$$

$$\times y_1^{\xi_1} x^{\mu_1 \xi_1} \quad (t-x)^{\upsilon_1 \xi_1}, \ldots, y_r^{\xi_r} x^{\mu_r \xi_r} (t-x)^{\upsilon_r \xi_r} d\xi_1, \ldots, d\xi_r dx$$

$$= e^{-zt} \int_0^t x^{\rho-1}(t-x)^{\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c,k)}{\left(c+\frac{1}{2},k\right)} \frac{a_k x^{\zeta k}(t-x)^{\eta k+u} z^u}{u!}$$

$$\times \frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} \emptyset_1(\xi_1)\ldots\emptyset_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)$$

$$\times\, {y_1}^{\xi_1}x^{\mu_1\xi_1}(t-x)^{\upsilon_1\xi_1},\ldots,{y_r}^{\xi_r}x^{\mu_r\xi_r}(t-x)^{\upsilon_r\xi_r}d\xi_1,\ldots,d\xi_r dx$$

$$[Now\ by\ using\ (9) the\ above\ result\ reduces\ to]$$

$$= e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u}\frac{(c,k)}{\left(c+\frac{1}{2},k\right)}\frac{a_k x^{\zeta k}(t-x)^{(\eta-1)k+u}}{(u-k)!}z^{u-k}$$
$$\times \frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} \emptyset_1(\xi_1)\ldots\emptyset_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)$$
$$\times\, {y_1}^{\xi_1}x^{\mu_1\xi_1}(t-x)^{\upsilon_1\xi_1},\ldots,{y_r}^{\xi_r}x^{\mu_r\xi_r}(t-x)^{\upsilon_r\xi_r}d\xi_1,\ldots,d\xi_r dx$$

Interchanging the order of integration and summation, we obtain

$$= e^{-zt}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} \emptyset_1(\xi_1)\ldots\emptyset_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)$$
$$\times\left[\int_o^t x^{\rho+\zeta r+\sum_{i=1}^{r}\mu_i\xi_i-1}(t-x)^{\sigma+(\eta-1)k+u+\sum_{i=1}^{r}\upsilon_i\xi_i-1}dx\right]$$
$$\times\, {y_1}^{\xi_1},\ldots,{y_r}^{\xi_r}d\xi_1,\ldots,d\xi_r$$

Where f(k) is given by (11)
On substituting x = ts in the inner x integral, the above expression reduces to

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} \emptyset_1(\xi_1)\ldots\emptyset_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)t^{\sum_{i=1}^{r}(\mu_i+\upsilon_i)\xi_i}$$
$$\times\left[\int_o^1 s^{\rho+\zeta k\ \sum_{i=1}^{r}\mu_i\xi_i-1}(1-s)^{\sigma+(\eta-1)k+u+\sum_{i=1}^{r}\upsilon_i\xi_i-1}ds\right]\quad {y_1}^{\xi_1},\ldots,{y_r}^{\xi_r}d\xi_1,\ldots,d\xi_r$$

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}$$
$$\times\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r} \emptyset_1(\xi_1)\ldots\emptyset_r(\xi_r)\psi(\xi_1,\ldots,\xi_r)\frac{\Gamma(\rho+\zeta k+\sum_{i=1}^{r}\mu_{i\xi_i})\Gamma(\sigma+(\eta-1)k+u+\sum_{i=1}^{r}\upsilon_{i\xi_i})}{\Gamma(\rho+\sigma+(\zeta+\eta-1)k+u+\sum_{i=1}^{r}(\mu_i+\upsilon_i)\xi_i\ )}$$
$$\times\ {y_1}^{\xi_1}t^{(\mu_1+\upsilon_1)\xi_1},\ldots,{y_r}^{\xi_r}t^{(\mu_r+\upsilon_r)\xi_r}d\xi_1,\ldots,d\xi_r$$

Finally, interpreting the contour integral by virtue of (5), we obtain

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}$$

$$\times H^{0,n+2:m_1,n_1;\ldots;m_r,n_r}_{p+2,q+1:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} y_1 t^{(\mu_1+\upsilon_1)}\\ \vdots \\ y_r t^{(\mu_r+\upsilon_r)}\end{matrix}\left|\begin{matrix}(1-\rho-\zeta k;\prod_{i=1}^{r}\mu_i),(1-\sigma-(\eta-1)k-u;\prod_{i=1}^{r}\upsilon_i)(a_j;\alpha_j^1,\ldots,\alpha_j^r)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};\ldots;(c_j^r,\gamma_j^r)_{1,p_r}\\ (b_j;\beta_j^1,\ldots,\beta_j^r)_{1,q},(1-\rho-\sigma-(\zeta+\eta-1)k-u;\prod_{i=1}^{r}\mu_i+\upsilon_i):(d_j^1,\delta_j^1)_{1,q_1};\ldots;(d_j^r,\delta_j^r)_{1,q_r}\end{matrix}\right.\right]\quad(12)$$

IV. PARTICULAR CASE:

(a) Putting r=2, t=1 in theorem (5) the new results may be realized to the H-function of two variables:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}\,{}_2F_1\left(a,b,c+\tfrac{1}{2};x^{\zeta}(1-x)^{\eta}\right){}_2F_1\left(c-a,c-b,c+\tfrac{1}{2};x^{\zeta}(1-x)^{\eta}\right)$$

$$\times H^{0,n;m_1,n_1;m_2,n_2}_{p,q;p_1,q_1;p_2,q_2}\left[\begin{matrix} y_1x^{\mu_1}(1-x)^{\upsilon_1} \\ \\ y_2x^{\mu_2}(1-x)^{\upsilon_2}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j,\beta_j^1,\beta_j^2)_{1,q}:(d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2}\end{matrix}\right.\right]dx$$

$$= e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}$$

$$\times \mathrm{H}^{0,\,n+2:\,m_1,\,n_1;\,m_2,\,n_2}_{p+2,\,q+1:\,p_1,\,q_1;\,p_2,\,q_2}\left[\begin{matrix} y_1 \\ \\ y_2\end{matrix}\left|\begin{matrix}(1-\rho-\zeta k;\mu_1,\mu_2),(1-\sigma-(\eta-1)k-u;\upsilon_1,\upsilon_2)(a_j,\alpha_j^1,\alpha_j^2)_{1,p}:(c_j^1,\gamma_j^1)_{1,p_1};(c_j^2,\gamma_j^2)_{1,p_2} \\ (b_j;\beta_j^1,\beta_j^2)_{1,q},(1-\rho-\sigma-(\zeta+\eta-1)k-u;\mu_1+\upsilon_1,\mu_2+\upsilon_2):(d_j^1,\delta_j^1)_{1,q_1};(d_j^2,\delta_j^2)_{1,q_2}\end{matrix}\right.\right] \quad (13)$$

(b) Putting n=p=q=0,r=1 and t=1 in theorem (5) the new results may be realized to the H-function of one variables:

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}e^{-zx}\,{}_2F_1\left(a,b,c+\tfrac{1}{2};x^{\zeta}(1-x)^{\eta}\right){}_2F_1\left(c-a,c-b,c+\tfrac{1}{2};x^{\zeta}(1-x)^{\eta}\right)$$

$$\times H^{m,n}_{p,q}\left[yx^{\mu}(1-x)^{\nu}\left|\begin{matrix}(c_j,\gamma_j)_{1,p} \\ (d_j,\delta_j)_{1,q}\end{matrix}\right.\right]dx$$

$$= e^{-z}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}$$

$$\times \mathrm{H}^{m,n+2}_{p+2,q+1}\left[y\left|\begin{matrix}(1-\rho-\zeta k;\mu),(1-\sigma-(\eta-1)k-u;\upsilon)(c_j,\gamma_j)_{1,p} \\ (d_j,\delta_j)_{1,q},(1-\rho-\sigma-(\zeta+\eta-1)k-u;\mu+\upsilon)\end{matrix}\right.\right] \quad (14)$$

V. CONCLUSION

The Multivariable H-function, Presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions such as Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, Mac-Robert's E-function, generalized hypergeometric function, Bessel function of first kind, modified Bessel function, Whittaker function, exponential function, binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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